THE GENERALIZED MATRIX VALUED HYPERGEOMETRIC EQUATION

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ABSTRACT. The matrix valued analog of the Euler's hypergeometric differential equation was introduced by Tirao in [1]. This equation arises in the study of matrix valued spherical functions and in the theory of matrix valued orthogonal polynomials. The goal of this paper is to extend naturally the number of parameters of Tirao's equation in order to get a generalized matrix valued hypergeometric equation. We take advantage of the tools and strategies developed in [1] to identify the corresponding matrix hypergeometric functions $_nF_m$. We prove that, if $n = m + 1$, this functions are analytic for $|z| < 1$ and we give a necesary condition for the convergence on the unit circle $|z| = 1$.

1. INTRODUCTION

The importance of the hypergeometric differential equation introduced by Euler in 1769 and the hypergeometric function was perceived by famous mathematicians as Gauss, Kummer and Riemman. Since then, many others found generalizations and applications. In fact many of the special functions that appear in mathematical physics, engineering and statistics are special cases of hypergeometric functions. Every second-order ordinary differential equation with three regular singular points, by placing the singularities in at 0, 1, and ∞, can be transformed into the hypergeometric differential equation

$$z(1 - z)f''(z) + (c - z(a + b + 1))f'(z) - abf(z) = 0,$$

where a, b and c are supposed to be complex numbers. If c is not an integer we can verify by a direct differentiation of the series that the unique solution of this equation, analytic and with value 1 at $z = 0$, is given by the hypergeometric function defined by

$$_2F_1\left(\begin{smallmatrix} a\,,\,b \\ c \end{smallmatrix}; z\right) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \text{ where } (w)_n = w(w + 1) \ldots (w + n - 1).$$

Recently, Tirao introduced in [1] the matrix valued analog of the hypergeometric differential equation

$$(1) \qquad z(1 - z)F''(z) + (C - z(A + B + 1))F'(z) - ABF(z) = 0,$$

where $A, B, C \in \mathbb{C}^{r \times r}$ and F denotes a function on $\mathbb{C}$ with values in $\mathbb{C}^r$. The corresponding matrix analog of the Gauss' hypergeometric function is given by

$$_2F_1\left(\begin{smallmatrix} A\,;\,B \\ C \end{smallmatrix}; z\right) = \sum_{n \geq 0} \left(\begin{smallmatrix} A\,;\,B \\ C \end{smallmatrix}\right)_n \frac{z^n}{n!},$$

where $\left({}^{A\,;\,B}_{C} \right)_0$ is the identity matrix and

$$\left({}^{A\,;\,B}_{C} \right)_{n+1} = (C+n)^{-1}(A+n)(B+n) \left({}^{A\,;\,B}_{C} \right)_n, \text{ for all } n \geq 0.$$

The matrix valued hypergeometric functions have connections with the theory of matrix valued spherical functions and with the theory of matrix valued orthogonal polynomials ([2],[3],[4]). In fact in [2] we prove that the matrix representations of irreducible spherical functions associated to the dual symmetric pairs $(\mathrm{SU}(2,1),\mathrm{U}(2))$ and $(\mathrm{SU}(3),\mathrm{U}(2))$ are given in terms of matrix hypergeometric functions.

The goal of this paper is to generalize the matrix valued hypergeometric equation (1) by extending naturally the number of parameters as it was done in the scalar case at the end of the nineteenth century. In this way we shall consider the following generalized matrix hypergeometric equation

$$(2) \quad z\frac{\mathrm{d}}{\mathrm{d}z} \left(z\frac{\mathrm{d}}{\mathrm{d}z} + B_1 - 1 \right) \left(z\frac{\mathrm{d}}{\mathrm{d}z} + B_2 - 1 \right) \dots \left(z\frac{\mathrm{d}}{\mathrm{d}z} + B_m - 1 \right) F(z)$$

$$- z \left(z\frac{\mathrm{d}}{\mathrm{d}z} + A_1 \right) \left(z\frac{\mathrm{d}}{\mathrm{d}z} + A_2 \right) \dots \left(z\frac{\mathrm{d}}{\mathrm{d}z} + A_n \right) F(z) = 0,$$

where $n, m \in \mathbb{Z}_{\geq 0}$, $A_1, \dots, A_n, B_1, \dots, B_m \in \mathbb{C}^{r \times r}$ and F denotes a complex function with values in $\mathbb{C}^r$. Observe that if we set $n = 2$ and $m = 1$ we obtain the Tirao's hypergeometric equation.

If no eigenvalue of $B_1, \dots, B_m$ is in the set $\{0, -1, -2, \dots\}$ we introduce the generalized matrix valued hypergeometric function in the following way

$$ {}_nF_m \left({}^{A_1;\,\dots\,;A_n}_{B_1;\,\dots;B_m}; z \right) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \left({}^{A_1;\,\dots\,;A_n}_{B_1;\,\dots;B_m} \right)_j,$$

where the symbol $\left({}^{A_1;\,\dots\,;A_n}_{B_1;\,\dots;B_m} \right)_j$ is defined in (8). We prove that this functions are solutions of the differential equation (2). Furthermore if $n = m + 1$ and $\{V_j\}_{j=1}^r$ is a basis of $\mathbb{C}^r$ then the set

$$\left\{ {}_{m+1}F_m \left({}^{A_1;\,\dots\,;A_{m+1}}_{B_1;\,\dots;B_m}; z \right) V_j \right\}_{j=1}^r,$$

is basis of the space of all solutions of the hypergeometric equation (2) analytic at $z = 0$ for $|z| < 1$.

Throughout this paper for any matrix $A \in \mathbb{C}^{r \times r}$ we denote by $\sigma(A)$ to the set of all eigenvalues of A and

$$\rho(A) = max\{Re(\lambda) : \lambda \in \sigma(A)\}, \quad \delta(A) = min\{Re(\lambda) : \lambda \in \sigma(A)\}.$$

For any $A \in \mathbb{C}^{r \times r}$, we shall consider the spectral norm $\| \bullet \|$ defined by

$$\|A\| = \max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } AA^*\}.$$

Let us assume that $(\sigma(B_1) \cup \sigma(B_2) \cup \dots \cup \sigma(B_m)) \cap (-\mathbb{N}_0) = \emptyset$ and that $B_1, \dots, B_m$ are unitarily equivalent to a diagonalizable matrix with real eigenvalues. If

$$\delta(B_1) + \dots + \delta(B_m) \geq \|A_1\| + \dots + \|A_{m+1}\|,$$

in Theorem 2.4 we prove that ${}_{m+1}F_m \left({}^{A_1;\,\dots\,;A_{m+1}}_{B_1;\,\dots;B_m}; z \right)$ is absolutely convergent on the unit circle $|z| = 1$.

2. On the generalized matrix valued hypergeometric equation

The aim of this section is to give a generalization of the matrix valued hypergeometric equation to the case of an arbitrary number of parameters. Let $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathbb{C}^{r \times r}$. We begin by considering the following matrix valued differential equation of degree $\max(n, m)$.

$$
\text{(3)} \quad z \frac{d}{dz} \left(z \frac{d}{dz} + B_1 - 1 \right) \left(z \frac{d}{dz} + B_2 - 1 \right) \ldots \left(z \frac{d}{dz} + B_m - 1 \right) F(z)
$$
$$
- z \left(z \frac{d}{dz} + A_1 \right) \left(z \frac{d}{dz} + A_2 \right) \ldots \left(z \frac{d}{dz} + A_n \right) F(z) = 0.
$$

Let us denote by Δ to the differential operator $z \frac{d}{dz}$. Then, in terms of such operator, the matrix valued differential equation (3) is given by

$$
\text{(4)} \quad \Delta(\Delta + B_1 - 1)(\Delta + B_2 - 1) \ldots (\Delta + B_m - 1)F(z)
$$
$$
= z(\Delta + A_1)(\Delta + A_2) \ldots (\Delta + A_n)F(z).
$$

We will seek for solutions which have the following form

$$
F(z) = z^p \sum_{j=0}^{\infty} F_j z^j,
$$

where $F_j \in \mathbb{C}^r$. First at all we observe that

$$
\Delta F = z \frac{d}{dz} \sum_{j=0}^{\infty} F_j z^{p+j} = \sum_{j=0}^{\infty} (p+j) F_j z^{p+j},
$$

and therefore we have

$$
(\Delta + A_i)F = \sum_{j=0}^{\infty} (A_i + p + j) F_j z^{p+j}, \quad (\Delta + B_k - 1)F = \sum_{j=0}^{\infty} (B_k + p + j - 1) F_j z^{p+j},
$$

for all $1 \leq i \leq n$ and $0 \leq k \leq m$. If we substitute the previous results in the differential equation (4) we obtain the following recursion relation for the $\mathbb{C}^r$-valued coefficients F_j of $F(z)$

$$
\text{(5)} \quad (p + j + 1)(B_1 + p + j) \ldots (B_m + p + j)F_{j+1}
$$
$$
= (A_1 + p + j) \ldots (A_n + p + j)F_j.
$$

If we set $j = -1$ in the recursion relation we obtain

$$
\text{(6)} \qquad\qquad p(B_1 + p - 1)(B_2 + p - 1) \ldots (B_m + p - 1)F_0 = 0.
$$

Therefore we have the following indicial equation

$$
\text{(7)} \qquad\qquad p^r \det(B_1 + p - 1) \det(B_2 + p - 1) \ldots \det(B_m + p - 1) = 0.
$$

Let $\beta_1^j, \ldots, \beta_r^j$ be the eigenvalues of B_j. Then the roots of the indicial equation are given by

$$
p = 0, 1 - \beta_1^1, \ldots, 1 - \beta_r^1, 1 - \beta_1^2, \ldots, 1 - \beta_r^2, \ldots, 1 - \beta_1^m, \ldots, 1 - \beta_r^m.
$$

The case $p = 0$ correspond to the analytic solutions of the differential equation (3).

First at all we will describe the analytc solutions of the equation (3). Let $\left(\begin{smallmatrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{smallmatrix}\right)_j$ be the symbol defined recursively by

(8)
$$\left(\begin{matrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{matrix}\right)_0 = I,$$

$$\left(\begin{matrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{matrix}\right)_{j+1} = (B_m + j)^{-1}\ldots(B_1 + j)^{-1}(A_1 + j)\ldots(A_n + j)\left(\begin{matrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{matrix};z\right)_j.$$

Definition 2.1. If $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathbb{C}^{r\times r}$ and no eigenvalue of $B_1, \ldots, B_m$ is in the set $\{0, -1, -2, \ldots\}$, then we define the funtion

(9)
$$ {}_nF_m\left(\begin{matrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{matrix};z\right) = \sum_{j=0}^{\infty}\frac{z^j}{j!}\left(\begin{matrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{matrix}\right)_j.$$

Theorem 2.2. *The function ${}_nF_m\left(\begin{smallmatrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{smallmatrix};z\right)$ is analytic for all $z \in \mathbb{C}$ if $n \leq m$ and it is analytic on $|z| < 1$ if $n = m + 1$. On the other hand if $n > m + 1$ the series is not absolutely convergent unless $|z| = 0$.*

Proof. Let $\|\cdot\|$ be any matrix norm on $\mathbb{C}^{r\times r}$. We observe that

$$\left\|{}_nF_m\left(\begin{matrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{matrix};z\right)\right\| = \left\|\sum_{j=0}^{\infty}\left(\begin{matrix} A_1;\ ...\ ;A_n \\ B_1;...;B_m \end{matrix}\right)_j\frac{z^j}{j!}\right\| \leq \sum_{j=0}^{\infty}\left\|\left(\begin{matrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{matrix}\right)_j\right\|\frac{|z|^j}{j!}.$$

Now we will prove that the power series $\sum_{j=0}^{\infty}\left\|\left(\begin{smallmatrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{smallmatrix}\right)_j\right\|\frac{|z|^j}{j!}$ is convergent. In this way we denote $a_j = \left\|\left(\begin{smallmatrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{smallmatrix}\right)_j\right\|\frac{|z|^j}{j!}$. Then we shall compute $\lim_{j\to\infty}\frac{a_{j+1}}{a_j}$. Observe that

$$\frac{a_{j+1}}{a_j} = \frac{|z|}{j+1}\left\|\left(\begin{matrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{matrix}\right)_{j+1}\right\|\left\|\left(\begin{matrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{matrix}\right)_j\right\|^{-1}$$

$$\leq \frac{|z|}{j+1}\left\|(B_m + j)^{-1}\ldots(B_1 + j)^{-1}(A_1 + j)\ldots(A_n + j)\right\|$$

$$= \frac{|z|j^{n-m-1}}{\frac{1}{j}+1}\left\|\left(1+\tfrac{B_m}{j}\right)^{-1}\ldots\left(1+\tfrac{B_1}{j}\right)^{-1}\left(1+\tfrac{A_1}{j}\right)\ldots\left(1+\tfrac{A_n}{j}\right)\right\|,$$

Provided that $n \leq m$, this expression tends to zero as $j \to \infty$ and thus the series is not absolutely convergent unless $z = 0$. If $n > m + 1$, the series is convergent for all z. On the other side, for $n = m + 1$ the series converges for $|z| < 1$. $\qquad\square$

It is worth noticing that if $F_0 \in \mathbb{C}^r$ then ${}_nF_m\left(\begin{smallmatrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{smallmatrix};z\right)F_0$ is a solution of the differential equation (3). In the next theorem we summarize our results which characterize the analytic solutions at $z = 0$ of the generalized hypergeometric equation

Theorem 2.3. *Let us assume that $(\sigma(B_1) \cup \sigma(B_2) \cup \ldots \cup \sigma(B_m)) \cap (-\mathbb{N}_0) = \emptyset$ and let $F_0 \in \mathbb{C}^r$ then $F(z) = {}_nF_m\left(\begin{smallmatrix} A_1;\ ...\ ;A_n \\ B_1;\ ...\ ;B_m \end{smallmatrix};z\right)F_0$ is a solution of the differential equation (3) such that $F(0) = F_0$. Conversely any solution F analytic at $z = 0$ is of this form.*

For any $A \in \mathbb{C}^{r \times r}$, we shall consider the spectral norm $\| \bullet \|$ defined by

$$\|A\| = \max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } AA^*\}.$$

Since the spectral norm is a unitarily invariant matrix norm we observe that $\|XAX^*\| = \|A\|$ for any $A \in \mathbb{C}^{r \times r}$ and any unitary matrix $X \in \mathbb{C}^{r \times r}$.

Theorem 2.4. *Let $A_1, \ldots, A_{m+1}, B_1, \ldots, B_m \in \mathbb{C}^{r \times r}$ such that $(\sigma(B_1) \cup \sigma(B_2) \cup \ldots \cup \sigma(B_m)) \cap (-\mathbb{N}_0)$ is empty and let $B_1, \ldots, B_m$ be unitarily equivalent to a diagonalizable matrix with real eigenvalues. If*

$$\delta(B_1) + \ldots + \delta(B_m) \geq \|A_1\| + \ldots + \|A_{m+1}\|,$$

then the matrix hypergeometric function $_{m+1}F_m \left(\begin{smallmatrix} A_1; \, \ldots \, ;A_{m+1} \\ B_1; \, \ldots \, ;B_m \end{smallmatrix} ; z \right)$ is absolutely convergent for $|z| = 1$.

Proof. Observe that since B_i is diagonalizable, there exists $X_i \in \mathbb{C}^{r \times r}$ such that $X_i B_i X_i^* = \Lambda_i$ where Λ_i is a diagonal matrix with real no zero entries. Then we have that

$$\|B_i^{-1}\| = \|X_i B_i^{-1} X_i^*\| = \|\Lambda_i\| = \rho(B_i^{-1}) = \frac{1}{\delta(B_i)},$$

and

$$\|(B_i + kI)^{-1}\| = \frac{1}{\delta(B_i + kI)} = \frac{1}{\delta(B_i) + k}.$$

By hypothesis, there exists a positive number λ such that

$$\sum_{i=1}^{m} \delta(B_i) - \sum_{i=1}^{m+1} \|A_i\| = 2\lambda,$$

and we have that

$$\left\| \frac{|z|^k}{k!} \left(\begin{smallmatrix} A_1; \, \ldots \, ;A_{m+1} \\ B_1; \, \ldots \, ;B_m \end{smallmatrix} \right)_k \right\| \leq \frac{|z|^k}{k!} \prod_{i=1}^{m} \left(\prod_{h=0}^{k-1} \|(B_i + hI)^{-1}\| \right) \prod_{j=1}^{m+1} (\|A_j\|)_k$$

$$= \frac{|z|^k}{k!} \prod_{i=1}^{m} \frac{1}{(\delta(B_i))_k} \prod_{j=1}^{m+1} (\|A_j\|)_k$$

$$\leq |z|^k \, k^{\sum_{i=1}^{m} \delta(B_i) - \sum_{i=1}^{m+1} \|A_i\| - 1} \prod_{i=1}^{m} \left(\frac{(k-1)! \, k^{\delta(B_i)}}{(\delta(B_i))_k} \right)$$

$$\times \prod_{i=1}^{m+1} \left(\frac{(\|A_i\|)_k}{(k-1)! \, k^{\|A_i\|}} \right).$$

If we set $|z| = 1$, then

$$\lim_{k \to \infty} k^{1+\lambda} \left\| \frac{1}{k!} \left(\begin{smallmatrix} A_1; \, \ldots \, ;A_{m+1} \\ B_1; \, \ldots \, ;B_m \end{smallmatrix} \right)_k \right\| \leq \lim_{k \to \infty} k^{-\lambda} \prod_{i=1}^{m} \left(\frac{(k-1)! \, k^{\delta(B_i)}}{(\delta(B_i))_k} \right)$$

$$\times \prod_{i=1}^{m+1} \left(\frac{(\|A_i\|)_k}{(k-1)! \, k^{\|A_i\|}} \right)$$

$$= 0 \cdot \prod_{i=1}^{m} \Gamma(\delta(B_i)) \prod_{j=1}^{m+1} \Gamma(\|A_j\|)^{-1} = 0.$$

Since the series $\sum_{k=1}^{\infty} \frac{1}{k^{1+\lambda}}$ converge if $\lambda > 0$, the Dirichlet's criterium of numerical series of positive numbers implies the absolute convergence on the unit circle for $|z| = 1$ of the matrix hypergeometric function $_{m+1}F_m \left(\begin{smallmatrix} A_1; \, \ldots \, ; A_{m+1} \\ B_1; \, \ldots \, ; B_m \end{smallmatrix} ; z \right)$.   □

Now we will concern ourselves with the non analytic solutions of the matrix equation (3). By taking into account the recursion relation (5) we introduce the following definition.

Definition 2.5. Let us assume that $A_1, \ldots, A_n, B_1, \ldots, B_m$ are $r \times r$ complex matrices and suppose that $-p \notin (\sigma(B_1) + \mathbb{N}_0) \cup \ldots \cup (\sigma(B_m) + \mathbb{N}_0) \cup \mathbb{N}$. Then we define the function

$$(10) \qquad _nF_m^p \left(\begin{smallmatrix} A_1; \, \ldots \, ; A_n \\ B_1; \, \ldots \, ; B_m \end{smallmatrix} ; z \right) = \sum_{j=0}^{\infty} \frac{z^j}{(p+1)_j} \left(\begin{smallmatrix} A_1+p; \, \ldots \, ; A_n+p \\ B_1+p; \, \ldots \, ; B_m+p \end{smallmatrix} \right)_j.$$

Observe that $p = 0$ implies that $_nF_m^p \left(\begin{smallmatrix} A_1; \, \ldots \, ; A_n \\ B_1; \, \ldots \, ; B_m \end{smallmatrix} ; z \right) = _nF_m \left(\begin{smallmatrix} A_1; \, \ldots \, ; A_n \\ B_1; \, \ldots \, ; B_m \end{smallmatrix} ; z \right)$.

If $\beta \in \sigma(B_1) \cup \cdots \cup \sigma(B_m)$ and $\beta \neq 1$ then $p = 1 - \beta$ is a nonzero solution of the indicial equation

$$p^r \det(B_1 + p - 1) \det(B_2 + p - 1) \ldots \det(B_m + p - 1) = 0.$$

Thus the kernel of the matrix $p(B_1 + p - 1) \ldots (B_m + p - 1)$ is nonempty and we can take a nonzero vector $F_\beta \in \ker(p(B_1 + p - 1) \ldots (B_m + p - 1))$. Then, because of the way it was constructed, the $\mathbb{C}^r$-valued function

$$F(z) = z^{1-\beta} {}_nF_m^{1-\beta} \left(\begin{smallmatrix} A_1; \, \ldots \, ; A_n \\ B_1; \, \ldots \, ; B_m \end{smallmatrix} ; z \right) F_\beta,$$

is a solution of the matrix generalized hypergeometric equation (3). We ressume this fact in the following theorem.

Theorem 2.6. *Let* $\beta \in \sigma(B_1) \cup \cdots \cup \sigma(B_m)$ *and* $\beta \neq 1$. *If* $\beta = 1 - p$ *and we assume that* $\beta \notin (\sigma(B_1) + \mathbb{N}) \cup \ldots \cup (\sigma(B_m) + \mathbb{N}) \cup \mathbb{N}$ *and* F_β *is a vector in* $\ker(p(B_1 + p - 1) \ldots (B_m + p - 1))$ *then*

$$F(z) = z^{1-\beta} {}_nF_m^{1-\beta} \left(\begin{smallmatrix} A_1; \, \ldots \, ; A_n \\ B_1; \, \ldots \, ; B_m \end{smallmatrix} ; z \right) F_\beta = \sum_{j=0}^{\infty} \frac{z^{p+j}}{(p+1)_j} \left(\begin{smallmatrix} A_1+p; \, \ldots \, ; A_n+p \\ B_1+p; \, \ldots \, ; B_m+p \end{smallmatrix} \right)_j F_\beta,$$

is a solution of the generalized hypergeometric equation (3).

Remark 2.7. In the case that $n = m + 1$, if the matrices B_i, $i = 1, \ldots, m$, are all diagonalizable, and the set of eigenvalues $\{\beta_i^j\}$ is such that $\beta_i^j \neq \beta_k^h$ if $(i, j) \neq (k, h)$, the set of all analytic solutions of (3) given in Theorem 2.3 and the set of all non-analytic solutions given in Theorem 2.6 forms a fundamental set of solutions of the hypergeometric equation (3).

3. AN EXAMPLE FROM THE REPRESENTATION THEORY

The importance of the study of the matrix valued hypergeometric equation, and its solutions, become evident at the light of its connections with the theory of spherical functions on any K-type on a Lie group and with the matrix valued orthogonal polynomials (See [2, 3, 4]). However the hypergeometric equation that appear in [2] is slightly different from equation (1) and has the form

$$(11) \qquad u(1-u)\frac{d^2 F(u)}{du^2} + (C - uU)\frac{dF(u)}{du} - VF(u) = 0,$$

for $r \times r$ complex matrices C, U and V and a $\mathbb{C}^r$-valued function F on $\mathbb{C}$. This differential equation cannot always be reduced to a hypergeometric equation

$$(12) \qquad z(1-z)F''(z) + (C - z(A+B+1))F'(z) - ABF(z) = 0,$$

because we need to be able to find two matrices A and B which are solutions of the equations $U = A + B + 1$, and $V = AB$, or equivalently we need to solve the matrix quadratic equation $B^2 + (1-U)B + V = 0$. This is not a trivial fact because a matrix quadratic equation may not have any solution. For example there is not any 2×2 complex matrix X such that

$$X^2 - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0.$$

Now we will concern ourselves in a particular case where it is posible to transform the equation (11) into the hypergeometric equation (12). Let $Q(\lambda) = \lambda^2 + \lambda(1 - U) + V$. Then $\det(Q(\lambda))$ is a the polynomial in λ of degree $2r$ and therefore there are exactly $2r$ solutions $\{\lambda_1, \ldots, \lambda_{2r}\}$ of the equation $\det(Q(\lambda)) = 0$. For each λ_i, we have that the matrix $Q(\lambda_i)$ is singular. Let us assume that we can choose r different eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_r}$ and r linearly independent vectors $p_{i_1}, \ldots, p_{i_r}$ such that $p_{i_k} \in \ker Q(\lambda_{i_k})$. Let Λ be the diagonal matrix whose j-th diagonal element is given by λ_{i_j} and let X be the $r \times r$ matrix whose j-th column is the vector p_{i_j}. Now we observe that

$$X\Lambda^2 + (1-U)X\Lambda - VX\Lambda = 0.$$

If we multiply the previous equation by X^{-1} on the right, we obtain that

$$(X\Lambda X^{-1})^2 + (1-U)X\Lambda X^{-1} - VX\Lambda X^{-1} = 0,$$

and therefore $B = X\Lambda X^{-1}$ is a solution of the matrix quadratic equation $B^2 + (1-U)B + V = 0$. We resume this discussion in the following theorem.

Theorem 3.1. *Let $C, U, V \in \mathbb{C}^{r \times r}$ be such that we can choose r different eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_r}$ and r linearly independent vectors $p_{i_1}, \ldots, p_{i_r}$ such that $p_{i_k} \in \ker Q(\lambda_{i_k})$. Let $\Lambda = \mathrm{diag}(\lambda_{i_j})$ and let $X = [p_{i_1}, \ldots, p_{i_r}]$, i.e. that the j-th column of X is p_{i_j}. Then if we let*

$$B = X\Lambda X^{-1}, \quad A = U - X\Lambda X^{-1} - 1,$$

we have that the differential equation (11) *is the following hypergeometric equation*

$$z(1-z)F''(z) + (C - z(A+B+1))F'(z) - ABF(z) = 0.$$

Example: Let us consider the differential equation

$$(13) \qquad u(1-u)\frac{d^2F(u)}{du^2} + (C - uU)\frac{dF(u)}{du} - VF(u) = 0,$$

where the coefficient matrices are

$$C = \sum_{i=0}^{\ell}(\beta + 1 + 2i)E_{ii} + \sum_{i=1}^{\ell}iE_{i,i-1}, \quad U = \sum_{i=0}^{\ell}(\alpha + \beta + \ell + i + 2)E_{ii},$$

$$V = \sum_{i=0}^{\ell}i(\alpha + \beta + i - k + 1)E_{ii} - \sum_{i=0}^{\ell-1}(\ell - i)(i + \beta - k + 1)E_{i,i+1},$$

with $\alpha, \beta > -1$, $0 < k < \beta + 1$ and $\ell \in \mathbb{N}$.

If we replace α by $m \in \mathbb{Z}_{\geq 0}$ and β by $n - 1$, this differential equation arises from the first few steps in the explicit determination of all matrix valued spherical functions associated to the n-dimensional projective space $P_n(\mathbb{C}) = \mathrm{SU}(n+1)/\mathrm{U}(n)$. Furthermore, together whith an appropiate choice of a matrix weight $W(u)$, it provides examples of families of Jacobi type matrix valued orthogonal polynomials (see [3] and [4]).

The goal is to use the content of Theorem 3.1 to find matrices A and B such that $U = A + B + 1$ and $V = AB$. First at all we observe that the matrix $Q(\lambda) = \lambda^2 + \lambda(1 - U) + V$ is an upper triangular matrix whose i-th diagonal entry is given by

$$\lambda^2 - \lambda(\alpha + \beta + \ell + i + 1) + i(\alpha + \beta + i - k + 1).$$

Therefore we have that $\det(Q(\lambda)) = \prod_{i=0}^{\ell}(\lambda^2 - \lambda(\alpha+\beta+\ell+i+1)+i(\alpha+\beta+i-k+1))$. Then we can choose λ_i as a solution of the cuadratic equation $\lambda^2 - \lambda(\alpha + \beta + \ell + i + 1) + i(\alpha + \beta + i - k + 1) = 0$ and a nonzero $v_i \in \ker Q(\lambda_i)$ for each $0 \leq i \leq \ell$. For simplicity we shall consider only the case in which $\lambda_i \neq \lambda_j$ for $i \neq j$. Observe that, since $Q(\lambda)$ is an upper triangular matrix, $\{v_i\}_{i=0}^{\ell}$ is a linearly independent set of vectors. Then if we set $X = [v_1, \ldots, v_\ell]$ and Λ as the diagonal matrix whose j-th diagonal element is given by $\Lambda_{jj} = \lambda_j$, we have that the differential equation (13) become equal to

$$z(1 - z)F''(z) + (C - z(A + B + 1))F'(z) - ABF(z) = 0,$$

where $A = U - X\Lambda X^{-1} - 1$ and $B = X\Lambda X^{-1}$.

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